General Relativity and Applications
1. From Special to General

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General Relativity is a Physical Theory

In principle, general relativity might be shown to be false.

Perhaps we are already seeing its breakdown (dark energy?).

It is (at least) a very good approximation to the truth.

To test it, we must first understand it.

Special Relativity

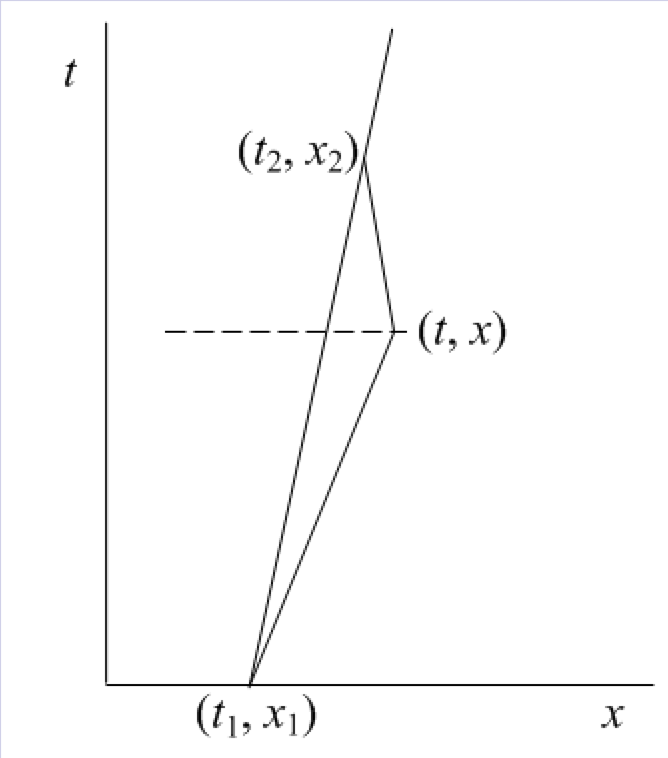
"Nothing can go faster than light."

~~E=ma²~~ ~~E=mb²~~ ~~E=mc²~~ $E = \dfrac{mc^2}{\sqrt{1 - v^2/c^2}}$

The laws of physics have the same form in all Minkowski coordinate systems (i.e., inertial reference frames).

Lorentz Transformations change coordinates from one Minkowski system to another.

Twin Paradox (Al-Qawsouni, 1634)



Which path from (t_1,x_1) to (t_2,x_2) is taken by a freely-falling astronaut?

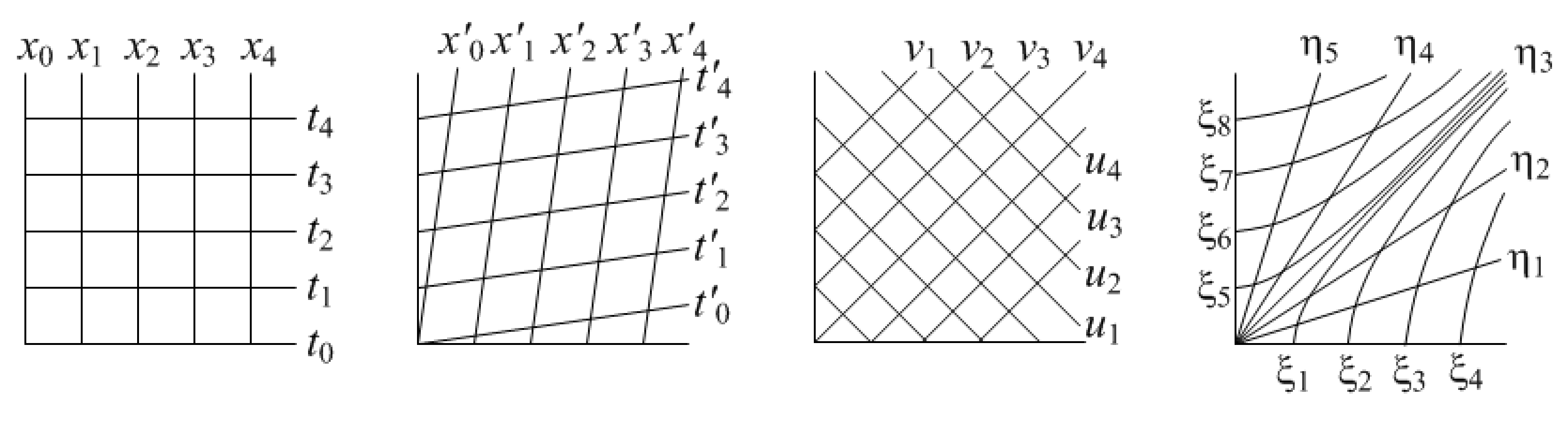
Which path is the longest?

Which astronaut has aged more?

$$(\text{proper time})^2 = (\Delta t)^2 - (\Delta x)^2$$

For convenience, choose units so that c=1.

Minkowski Spacetime: Coordinates are merely labels.



Lab frame	Lorentz boosted	Light cone	Rindler (accelerated)

By convention, special relativity is usually formulated using only the first two systems.

Points and curves in spacetime

Point x^μ, μ = 0,1,2,3: coordinates (e.g., $x^0=t, x^1=x, x^2=y, x^3=z$).

Spacetime distance between two nearby points:

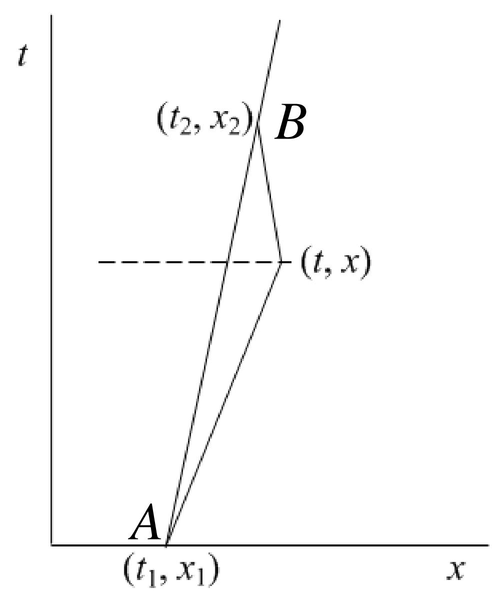
$$(ds)^2 = -(d\tau)^2 = -(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$

Spacetime curve:

$$x^\mu(\lambda) \ , \ \text{e.g.} \ \{t(\lambda), x(\lambda), y(\lambda), z(\lambda)\}$$

Tangent vector to a curve: $V^\mu = \dfrac{dx^\mu}{d\lambda}$

Extremal curves: geodesics



A curve $x^\mu(\lambda)$ has length (proper time)

$$\begin{aligned} T[x^\mu(\lambda)] &= \sum_A^B \left[(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2\right]^{1/2} \\ &= \int_A^B \left(- \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} d\lambda \\ &= \int_A^B L(x^\mu, dx^\mu/d\lambda)\, d\lambda \end{aligned}$$

$$\text{where } \eta_{\mu\nu} = \begin{cases} -1, & \mu = \nu = 0, \\ +1, & \mu = \nu = 1, 2, 3, \\ 0. & \mu \neq \nu \ . \end{cases}$$

Calculus of variations: Euler-Lagrange equation

$$\frac{d}{d\lambda}\left[\frac{\partial L}{\partial(dx^\mu/d\lambda)}\right] - \frac{\partial L}{\partial x^\mu} = 0 \ \Rightarrow \ \frac{d^2 x^\mu}{d\tau^2} = 0 \ , \ d\tau = L d\lambda \ .$$

Einstein Equivalence Principle

1. Motion in a gravitational field is locally indistinguishable from motion in an accelerated frame (Weak Equivalence Principle).

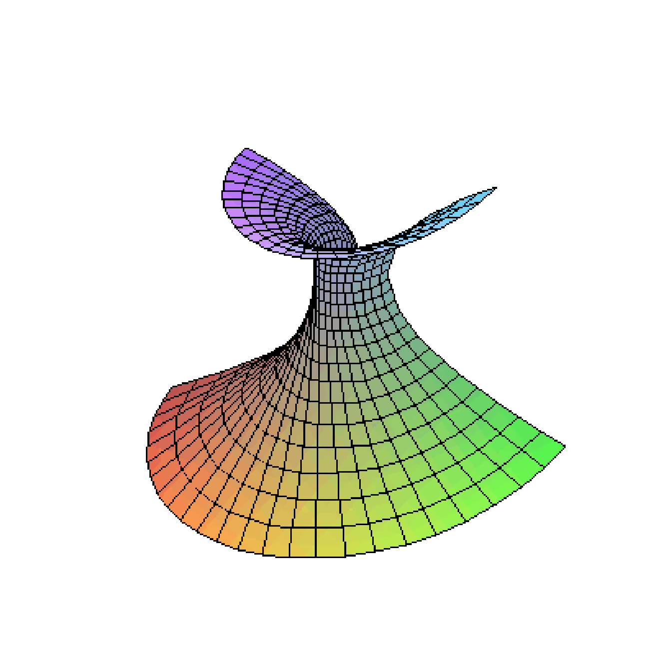
2. The outcome of any local non-gravitational experiment is independent of the velocity or spacetime position of the inertial reference frame in which it is performed (Local Lorentz and Poincaré Invariance).

Implications of the Equivalence Principle

Accelerated frames in flat spacetime are described by curvilinear coordinates. $\therefore$ Theories of gravity are formulated in curvilinear coordinates.

Non-constant gravitational fields exert tides which limit the size over which reference frames are locally inertial. $\therefore$ Global Minkowski coordinate systems do not exist in the presence of gravity.

Piecing Together a General Spacetime



Riemannian Manifold =
Smooth space approximated locally by flat sections.

The local geometry of a Riemannian Manifold is determined completely by the distance formula (line element or metric).

$$(ds)^2 = -(d\tau)^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu}(x)(dx^\mu)(dx^\nu)$$

From special to general relativity: $\boxed{\eta_{\mu\nu} \to g_{\mu\nu}(x)}$.

Geodesics in a general spacetime

Weak Equivalence Principle: Freely-falling bodies follow extremal curves (geodesics).

Calculus of variations: Euler-Lagrange

$$\frac{d}{d\lambda}\left[\frac{\partial L}{\partial(dx^\mu/d\lambda)}\right] - \frac{\partial L}{\partial x^\mu} = 0 \;\; \Rightarrow \;\; \boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0},$$

where $d\tau = L d\lambda$ and $g_{\mu\nu}\Gamma^\nu{}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\mu\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\mu}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right)$.

Einstein summation convention: implied sum on paired upper+lower indices

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General Relativity

“Spacetime tells matter how to move; matter tells spacetime how to curve.” (Wheeler)

The laws of physics have the same form in all coordinate systems.

Gravity is a fictitious force (like Coriolis).

Gravity is not a force; it is a manifestation of spacetime curvature.

General Relativity

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Gravity is a fictitious force (like Coriolis).

Gravity <u>is</u> a force <u>and</u> it is a manifestation of spacetime curvature. (Force/Geometry duality)

Gravity as Spacetime Curvature 1

Weak-field limit:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \quad |h_{\mu\nu}| \ll 1$$

Rename metric perturbations:

$$h_{00} = -2\Phi \ , \quad h_{0i} = w_i \ , \quad h_{ij} = 2(-\Psi\delta_{ij} + s_{ij}) \ ,$$

where $\delta^{ij} s_{ij} = 0$ GPS must correct for Φ!

Four-velocity $V^\mu = dx^\mu/d\tau$:

$$V^0 = \gamma(1 - \Phi + \underline{v} \cdot \underline{w}) \ , \quad V^i = \gamma(\delta^i{}_j - \tfrac{1}{2}h^i{}_j)v^j$$

where $i, j \in \{1, 2, 3\}$, $\underline{v} \cdot \underline{w} \equiv v^i w_i$, and $\gamma \equiv \dfrac{1}{\sqrt{1 - v^2}}$

Gravity as Spacetime Curvature 2

Geodesic equation:

$$\frac{dV^\mu}{d\tau} + \Gamma^\mu{}_{\alpha\beta} V^\alpha V^\beta = 0$$

Components (note $\partial_i \equiv \partial/\partial x^i$):

$$\begin{aligned}
\frac{dV^0}{d\tau} &= -\gamma^2 \left[\partial_t \Phi + 2v^i \partial_i \Phi + \left(\tfrac{1}{2}\partial_t h_{ij} - \partial_i w_j\right) v^i v^j\right] \\
\frac{dV^i}{d\tau} &= -\gamma^2 \Big[\partial_i \Phi + \partial_t w_i + (\partial_j w_i - \partial_i w_j + \partial_t h_{ij}) v^j \\
&\qquad + (\partial_k h_{ij} - \tfrac{1}{2}\partial_i h_{jk}) v^j v^k\Big]
\end{aligned}$$

~~This is so messy that it’s better not to try to understand gravity.~~

Why do we use vectors or index notation?

1. Equations like $\frac{d\vec{v}}{dt} = \frac{1}{m}\vec{F}$ (vector notation) or $\frac{dv_i}{dt} = \frac{1}{m}F_i$ are simpler to write than

$$\frac{dv_x}{dt} = \frac{1}{m}F_x \ , \quad \frac{dv_y}{dt} = \frac{1}{m}F_y \ , \quad \frac{dv_z}{dt} = \frac{1}{m}F_z \ .$$

2. Vectors group together objects that have a physical relationship.

3. Vector equations are valid independently of the coordinate system or basis which one uses.

Summary

Special Relativity unites time and space in spacetime. Describing motion requires points, curves, and vectors in spacetime.

Nature does not impose coordinates; laws of motion must hold independently of our choices (relativity principle).

Matter and energy curve spacetime.

Gravitational forces arise from spacetime curvature, which causes parallel lines to converge or diverge (Euclid was wrong!).