2. PRELIMINARIES

Laplace Transforms, Moment Generating Functions and Characteristic Functions

Laplace Transforms, Moment Generating Functions and Characteristic Functions

2.1. Definitions:

Let $\varphi(t)$ be defined on real line.

a. Moment Generating Function (MGF)

$$MGF = \int_{-\infty}^{\infty} e^{\theta t}\varphi(t)dt$$

which exists if $\int_{-\infty}^{\infty} |e^{\theta t}\varphi(t)dt| < \infty$

b. Characteristic Function (CF)

$$CF = \int_{-\infty}^{\infty} e^{iyt}\varphi(t)dt \qquad (i^2 = -1).$$

which exists if $\int_{-\infty}^{\infty} |e^{iyt}\varphi(t)|dt < \infty$

Since $e^{iyt} = \cos\ yt + i\ \sin yt, \quad |e^{iyt}| = 1$

CF exists if $\int_{-\infty}^{\infty} |\varphi(t)|dt < \infty$

c. Laplace Transform

Let $\varphi(t)$ be defined on $[0, \infty)$ and assume

$$\int_0^\infty e^{-x_0 t}\varphi(t)dt < \infty$$

for some value of $x_0 \geq 0$.

Characteristic function of $e^{-x_0 t}\varphi(t)$ is

$$CF = \int_0^\infty e^{-iyt}e^{-x_0 t}\varphi(t)dt \quad \text{(sign of } iyt \text{ could be + or -).}$$

$$CF = \int_0^\infty e^{-(x_0+iy)t}\varphi(t)dt$$

$$s = x_0 + iy$$

$$CF = \varphi^*(s) = \int_0^\infty e^{-st}\varphi(t)dt \ \text{ for } R(s) \geq x_0$$

$\varphi^*(s)$ is Laplace Transform of $\varphi(t)$.

$$\boxed{\varphi^*(s) = \int_0^\infty e^{-st}\varphi(t)dt}$$

Since $\varphi(t)$ defined on $[0, \infty)$, for $\varepsilon \geq 0$

$$\varphi^*(s) = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty e^{-st}\varphi(t)dt = \int_{0^+}^\infty e^{-st}\varphi(t)dt.$$

Often a script L is used to denote a LaPlace transform; i.e.

$$\boxed{\mathcal{L}\left\{\varphi(t)\right\} = \varphi^*(s)}$$

Suppose $f(t)$ is a pdf on $[0, \infty)$,

$$\boxed{f^*(s) = \int_0^\infty e^{-st}f(t)dt}, \;\; R(s) \geq 0$$

Laplace transform is appropriate for non-negative r.v.

If the cdf is $F(t) = \int_0^t f(x)dx = \int_0^t dF(x)$

$$\boxed{f^*(s) = \int_0^\infty e^{-st} dF(t)}$$

is Laplace-Stieltjes Transform.

Note: $E(e^{-sT}) = f^*(s)$

$$f^*(0) = 1, \quad 0 \leq f^*(s) \leq 1$$

2.2 Theorems on Laplace Transforms (LT)

a. Uniqueness Theorem.

Distinct probability distributions have distinct Laplace Transforms

b. Continuity Theorem

For $n = 1, 2, \ldots$, let $\{F_n(t)\}$ be a sequence of $cdf's$ such that $F_n \rightarrow F$.

Define $\{f_n^*(s)\}$ as the sequence of LT such that $\mathcal{L}\,\{f_n(t)\} = f_n^*(s)$ and define $f^*(s) = \int_0^\infty e^{-st} dF(t)$.

Then $f_n^*(s) \rightarrow f^*(s)$ and conversely if $f_n^*(s) \rightarrow f^*(s)$, then $F_n(t) \rightarrow F(t)$

c. <u>Convolution Theorem</u>

If T_1, T_2 are independent, non-negative r.v. with p.d.f $f_1(t), f_2(t)$ then the pdf of $T = T_1 + T_2$ is

$$f(t) = \int_0^\infty f_1(\tau) f_2(t-\tau) d\tau$$

and $\mathcal{L}\left\{f(t)\right\} = f_1^*(s) f_2^*(s)$

In general if $\{T_i\}$ $i = 1, 2, \ldots, n$ are independent non-negative r.v., then the Laplace transform of the pdf of $T = T_1 + \cdots + T_n$ is

$$\mathcal{L}\left\{f(t)\right\} = \prod_{i=1}^{n} f_i^*(s)$$

d. Moment Generating Property

Suppose all moments exist.

$$\begin{aligned} f^*(s) = \int_0^\infty e^{-st} f(t)dt &= \int_0^\infty \left\{ \sum_{n=0}^{\infty} (-1)^{\,n} \frac{s^n t^n}{n!} \right\} f(t)dt \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{s^n}{n!} \int_0^\infty t^n f(t)dt \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{s^n}{n!} m_n \ \text{ where } m_n = E(T^n) \end{aligned}$$

e. <u>Inversion Theorem</u>

Knowledge of $f^*(s)$. The inversion formula is written

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} f^*(s)ds$$

where the integration is in the complex plane and c is an appropriate constant. (It is beyond the scope of this course to discuss the inversion formula in detail).

<u>Notation</u>: $\mathcal{L}\left\{f(t)\right\} = f^*(s)$

$$f(t) = \mathcal{L}^{-1}\{f^*(s)\}$$

where $\mathcal{L}^{-1}\ \{\ \ \}$ is referred to as the "Inverse Laplace Transform".

Example: Exponential Distribution

$$\begin{aligned}
f(t) &= \lambda e^{-\lambda t} \text{ for } t \geq 0 \\
E(T) &= 1/\lambda, \;\; V(t) = 1/\lambda^2. \\
f^*(s) &= \int_0^\infty e^{-st}\lambda e^{-\lambda t} dt = \lambda \int_0^\infty e^{-(s+\lambda)t} dt \\
&= \left(\frac{\lambda}{\lambda+s}\right) \int_0^\infty (\lambda+s) e^{-(\lambda+s)t} dt = \frac{\lambda}{\lambda+s} \\
f^*(s) &= \frac{\lambda}{\lambda+s} = \frac{1}{1+\frac{s}{\lambda}} = \sum_{n=0}^\infty (-1)^n \left(\frac{s}{\lambda}\right)^n \\
&= \sum_{n=0}^\infty (-1)^n \frac{s^n}{n!} \left(\frac{n!}{\lambda^n}\right) \\
\Rightarrow & \;\; m_n = E(T^n) = \frac{n!}{\lambda^n}
\end{aligned}$$

$$\mathcal{L}\{\lambda e^{-\lambda t}\} = \frac{\lambda}{\lambda + s}, \qquad \mathcal{L}^{-1}\{\frac{\lambda}{\lambda + s}\} = \lambda e^{-\lambda t}$$

$$\mathcal{L}\{e^{-\lambda t}\} = (\lambda + s)^{-1}, \quad \mathcal{L}^{-1}\{(\lambda + s)^{-1}\} = e^{-\lambda t}$$

Note:

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} f^*(s)ds &= f(t) \\
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\left(\frac{\lambda}{\lambda + s}\right)ds &= \lambda e^{-\lambda t} \\
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}(\lambda + s)^{-1}ds &= e^{-\lambda t} \\
\mathcal{L}^{-1}\{(\lambda + s)^{-1}\} &= e^{-\lambda t}
\end{aligned}$$

$$\mathcal{L}^{-1}\{(\lambda+s)^{-1}\} = e^{-\lambda t}$$

Differentiating w.r. to λ

$$\mathcal{L}^{-1}\{(\lambda+s)^{-2}\} = \; te^{-\lambda t}$$

again $\quad \mathcal{L}^{-1}\{2(\lambda+s)^{-3}\} = \; t^2 e^{-\lambda t}$

$\vdots$

$(n-1)$ times $\quad \mathcal{L}^{-1}\{(n-1)!(\lambda+s)^{-n}\} = \; t^{n-1}e^{-\lambda t}$

$$\Rightarrow \qquad \mathcal{L}\left\{\frac{t^{n-1}e^{-\lambda t}}{\Gamma(n)}\right\} = (\lambda+s)^{-n}, \quad \Gamma(n) = (n-1)!$$

and $\quad \mathcal{L}\left\{\dfrac{\lambda(\lambda t)^{n-1}e^{-\lambda t}}{\Gamma(n)}\right\} = \; \lambda^n/(\lambda+s)^n$

Note: $\dfrac{\lambda(\lambda t)^{n-1}e^{-\lambda t}}{\Gamma(n)}$ is gamma distribution.

If $\{T_i\}$ $i = 1, 2, \ldots, n$ are independent non-negative identically distributed r.v. following an exponential distribution with parameter λ and $T = T_1 + \ldots + T_n$

$$f^*(s) = \prod_1^n f_i^*(s) = \left(\frac{\lambda}{\lambda + s}\right)^n$$

Example: Suppose T_1, T_2 are independent non-negative random variables following exponential distributions with parameters $\lambda_1,\ \lambda_2\ \ (\lambda_1 \neq \lambda_2)$. What is distribution of $T = T_1 + T_2$?

$$f^*(s) = f_1^*(s) f_2^*(s) = \left(\frac{\lambda_1}{\lambda_1 + s}\right)\left(\frac{\lambda_2}{\lambda_2 + s}\right).$$

$f^*(s)$ can be written as a partial fraction; i.e.

$$f^*(s) \quad = \quad \frac{A_1}{(\lambda_1 + s)} + \frac{A_2}{(\lambda_2 + s)} = \frac{A_1(\lambda_2 + s) + A_2(\lambda_1 + s)}{(\lambda_1 + s)(\lambda_2 + s)}$$

where $$A_1 \quad = \quad \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1},\ \ A_2 = -A_1$$

Since $\mathcal{L}^{-1}\{(\lambda+s)^{-1}\} = e^{-\lambda t}$

$$\begin{aligned} f(t) = \quad \mathcal{L}^{-1}\{f^*(s)\} &= A_1\mathcal{L}^{-1}\{(\lambda_1+s)^{-1}\} + A_2\mathcal{L}^{-1}\{(\lambda_2+s)^{-1}\} \\ &= A_1 e^{-\lambda_1 t} + A_2 e^{-\lambda_2 t} \\ &= \boxed{\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]} \end{aligned}$$

Suppose T_1, T_2, T_3 are non-negative independent r.v. so that T_1 is exponential with parameter λ_1 and T_2, T_3 are exponential each with parameter λ_2. Find pdf of $T = T_1 + T_2 + T_3$.

$$f^*(s) = \left(\frac{\lambda_1}{\lambda_1+s}\right)\left(\frac{\lambda_2}{\lambda_2+s}\right)^2 = \frac{A_1}{\lambda_1+s} + \frac{B_1}{\lambda_2+s} + \frac{B_2}{(\lambda_2+s)^2}$$

$$\boxed{f(t) = A_1 e^{-\lambda_1 t} + B_1 e^{-\lambda_2 t} + B_2 t e^{-\lambda_2 t}}$$

Homework:

Show $A_1 = (\lambda_1 - \lambda_2)^{-2}\lambda_1\lambda_2^2$

$$B_1 = -\left[(\lambda_1 - \lambda_2)^{-1} + (\lambda_1 - \lambda_2)^{-2}\right]\lambda_1\lambda_2^2$$

$$B_2 = (\lambda_1 - \lambda_2)^{-1}\lambda_1\lambda_2^2$$

In general if $f_i^*(s) = (\lambda_i/\lambda_i + s)\ \ i = 1, 2, \dots, n$ and $f^*(s) = \prod_{i=1}^{n} f_i^*(s)$, then f(t) is called the Erlangian distribution. (Erlang was a Danish telephone engineer who used this distribution to model telephone calls).

2.3 Operations on Laplace Transforms

$$\begin{aligned}\mathcal{L}\{e^{-\lambda t}\} &= (\lambda + s)^{-1} \\ \mathcal{L}\{t^{n-1}e^{-\lambda t}\} &= \Gamma(n)/(\lambda + s)^n\end{aligned}$$

Setting $\lambda = 0$ in above

$$\begin{aligned}\mathcal{L}\{1\} &= 1/s \\ \mathcal{L}\{t^{n-1}\} &= \Gamma(n)/s^n\end{aligned}$$

Homework: Prove the following relationships

1. $\mathcal{L}\left\{\int_0^t \varphi(x)dx\right\} = \dfrac{\varphi^*(s)}{s}$

2. $\mathcal{L}\left\{\varphi'(t)\right\} = s\varphi^*(s) - \varphi(0^+)$

3. $\mathcal{L}\left\{\varphi^{(r)}(t)\right\} = s^r\varphi^*(s) - s^{r-1}\varphi(0^+) - s^{r-2}\varphi'(0^+) - \ldots - \varphi^{(0)}(0^+)$

Prove (1) and (2) using integration by parts

Prove (3) by successive use of (2).

Suppose $f(t)$ is a pdf for a non-negative random variable and

$$F(t) = \int_0^t f(x)dx, \;\; Q(t) = \int_t^\infty f(x)dx$$

Since $\mathcal{L}\left\{\int_0^t \varphi(x)dx\right\} = \dfrac{\varphi^*(s)}{s}$

$$\boxed{\mathcal{L}\left\{F(t)\right\} = F^*(s) = \frac{f^*(s)}{s}}$$

Since $Q(t) = 1 - F(t)$

$$\boxed{\mathcal{L}\left\{Q(t)\right\} = Q^*(s) = \frac{1}{s} - F^*(s) = \frac{1 - f^*(s)}{s}}$$

2.4 Limit Theorems

$$\boxed{\lim_{t\to 0^+} \varphi(t) = \lim_{s\to\infty} s\varphi^*(s)}$$

Proof:

$$\mathcal{L}\left\{\varphi'(t)\right\} = s\varphi^*(s) - \varphi(0^+)$$

$$\lim_{s\to\infty} {\varphi'}^*(s) = \lim_{s\to\infty}[s\varphi^*(s) - \varphi(0^+)]$$

But $\lim_{s\to\infty} \int_0^\infty e^{-st}\varphi'(t)dt = 0$

$$\therefore \lim_{s\to\infty}[s\varphi^*(s) - \varphi(0^+)] = 0$$

$$\boxed{\lim_{t\to\infty} \varphi(t) = \lim_{s\to 0} s\varphi^*(s)}$$

<u>Proof:</u>

$$\lim_{s\to 0} \mathcal{L}\{\varphi'(t)\} = \lim_{s\to 0} \int_0^\infty e^{-st}\varphi'(t)dt = \int_0^\infty \varphi'(t)dt$$

$$= \lim_{t\to\infty} \int_0^t \varphi'(x)dx = \lim_{t\to\infty} [\varphi(t) - \varphi(0^+)].$$

Since $\mathcal{L}\{\varphi'(t)\} = s\varphi^*(s) - \varphi(0^+)$

$$\lim_{s\to 0} [s\varphi^*(s) - \varphi(0^+)] = \lim_{t\to\infty} [\varphi(t) - \varphi(0^+)]$$

$$\Rightarrow \lim_{s\to 0} s\varphi^*(s) = \lim_{t\to\infty} \varphi(t)$$

2.5. Dirac Delta Function

Sometimes it is useful to use the Dirac delta function. It is a "strange" function and has the property.

$$\delta(t) = \begin{cases} 0 \text{ for } t \neq 0 \\ \infty \quad t = 0 \end{cases}$$

Define

$$U(t) = \begin{cases} 1 \text{ if } t > 0 \\ 0 \quad t \leq 0 \end{cases}$$

Let $h > 0$

$$\delta(t, h) = \frac{U(t+h) - U(t)}{h} = \begin{cases} 0 & \text{if } t > 0 \\ \dfrac{1}{h} & \text{if } -h < t \leq 0 \\ 0 & \text{if } t \leq -h \end{cases}$$

Define

$$\delta(t) = \lim_{h\to 0} \delta(t,h) = \lim_{h\to 0} \frac{U(t+h) - U(t)}{h} = \begin{cases} 0 & \text{for } t \neq 0 \\ \infty & \text{for } t = 0 \end{cases}$$

Consider

$$\int_0^\infty \varphi(x)\delta(t-x)dx$$

$$= \int_0^\infty \varphi(x) \lim_{h\to 0} \left[\frac{U(t-x+h) - U(t-x)}{h}\right] dx$$

$$= \lim_{h\to 0} \frac{1}{h} \left\{\int_0^\infty \varphi(x)U(t-x+h)dx - \int_0^\infty \varphi(x)U(t-x)dx\right\}$$

Since $U(t-x+h) = \begin{cases} 1 & t-x+h > 0 \\ 0 & \text{otherwise} \end{cases}$

$$\int_0^\infty \varphi(x)\delta(t-x)dx$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{\int_0^{t+h} \varphi(x)dx - \int_0^t \varphi(x)dx\right\}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{\int_t^{t+h} \varphi(x)dx\right\} = \lim_{h\to 0}\left\{\frac{\varphi(t+\theta h)h}{h}\right\} = \varphi(t)$$

(by mean value theorem)

for some value of θ $0 < \theta < 1$.

$$\therefore \boxed{\int_0^\infty \varphi(x)\delta(t-x)dx = \varphi(t)}$$

Similarly $\varphi(t) = \int_0^\infty \varphi(t-x)\delta(x)dx$

$$\boxed{\varphi(t) = \int_0^\infty \varphi(x)\delta(t-x)dx = \int_0^\infty \varphi(t-x)\delta(x)dx}$$

Special Cases

$\varphi(x) = 1$ for all x

$$1 = \int_0^\infty \delta(t-x)dx = \int_0^\infty \delta(x)dx \text{ for all } t$$

$$\Rightarrow \boxed{\int_0^\infty \delta(x)dx = 1}$$

Suppose $\varphi(x) = e^{-sx}$. Then

$$\int_0^\infty e^{-sx}\delta(x-t)dx = e^{-st}$$

If $t = 0$, $\boxed{\int_0^\infty e^{-sx}\delta(x)dx = \mathcal{L}\{\delta(x)\} = 1}$

Example 1 Consider a non-negative random variable T such that $p_n = P\{T = t_n\}$. Define

$$\boxed{f(t) = \sum_{n=1}^{\infty} p_n \delta(t - t_n)}$$

$$\int_0^{\infty} f(t)dt = \sum_{n=1}^{\infty} p_n \int_0^{\infty} \delta(t - t_n)dt = \sum_{1}^{\infty} p_n = 1$$

$$\boxed{F(t_n) = P\{T \le t_n\} = \sum_{j=1}^{\infty} p_j U(t_{n+1} - t_j) = \int_0^{t_n} f(t)dt = \sum_{j \le n} p_j}$$

Note: If $t_j \ < \ t_{n+1} \ \ \Rightarrow \ \ t_j \ \le \ t_n$

$$f^*(s) = \int_0^{\infty} e^{-st} f(t)dt = \sum_{n=1}^{\infty} p_n \int_0^{\infty} e^{-st} \delta(t - t_n)dt = \sum_{n=1}^{\infty} p_n e^{-st_n}$$

Example 2

Consider a random variable T such that $p = P\{T = 0\}$ but for $T > 0$

$$P\{t_1 < T \leq t_2\} = \int_{t_1}^{t_2} q(x)dx$$

$$\begin{aligned} \Rightarrow \quad f(t) &= p\delta(t) + (1-p)q(t) \\ f^*(s) &= p + (1-p)q^*(s). \end{aligned}$$

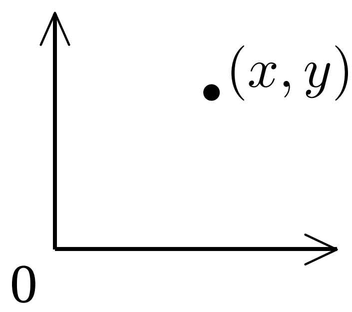
Another way of formulating this problem is to define the random variable T by

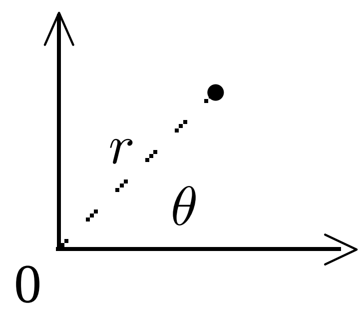
$$T = \begin{cases} 0 & \text{with probability } p \\ t & \text{with pdf } q(t) \text{ for } t > 0 \end{cases}$$

This is an example of a random variable having both a discrete and continuous part.

2.6 Appendix

ELEMENTS OF COMPLEX NUMBERS





Rectangular Coordinates

Polar Coordinates

r = Modulus

θ = Amplitude

$$x = r\cos\theta, \qquad y = r\sin\theta$$

Complex no. representation: $z = x + iy, \;\; r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1} \frac{x}{y}$

$|z|$ = absolute value= r $\quad$ i: to be determined

$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

Suppose $z_j = a_j + ib_j$

Addition/Subtraction: $z_1 \pm z_2 \Rightarrow (a_1 \pm a_2) + i(b_1 \pm b_2)$

Multiplication:

$$z_j = r_j e^{i\theta_j}, \quad z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \quad z_1 z_2 \Rightarrow r = r_1 r_2, \quad \theta = \theta_1 + \theta_2$$

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) = x + iy$$

$$x = r_1 r_2 \cos(\theta_1 + \theta_2) = r_1 r_2(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) = a_1 a_2 - b_1 b_2$$

$$y = r_1 r_2 \sin(\theta_1 + \theta_2) = r_1 r_2(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2) = b_1 a_2 + a_1 b_2$$

$$\Rightarrow z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = a_1 a_2 + i[a_1 b_2 + a_2 b_1] + i^2[b_1 b_2]$$

If $i^2 = -1 \quad Re(z_1 z_2) = a_1 a_2 - b_1 b_2$

$$i, i^2 = -1, \quad i^3 = -i, \quad i^4 = 1$$

$$\Rightarrow i^{4n+1} = i, \quad i^{4n+2} = -1, \quad i^{4n+3} = -i, \quad i^{4n} = 1$$

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Exponential function

$$
\begin{aligned}
e^{\theta} &= \sum_{0}^{\infty} \frac{\theta^n}{n!} \\
e^{i\theta} &= 1 + i\theta + \frac{i^2\theta^2}{2} + \frac{i^3\theta^3}{3!} + \ldots \\
&= (1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \ldots) \\
e^{i\theta} &= \cos\theta \quad + i\sin\theta, \\
e^{-i\theta} &= \cos\theta - i\sin\theta, \quad \cos(-\theta) = \cos\theta; \quad \sin -\theta = -\sin\theta \\
& \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \\
& \cos 0 = 1, \quad \sin 0 = 0
\end{aligned}
$$

Since $e^{i\theta} = \cos\theta + i\sin\theta; z = re^{i\theta}$

If $z = e^{x+iy} = e^x e^{iy}$: e^x : modulus, e^y : amplitude

$z_1 z_2 = e^{x_1+iy_1} \cdot e^{x_2+iy_2} = e^{x_1+x_2+i(y_1+y_2)}$

MOMENT GENERATING AND CHARACTERISTIC FUNCTIONS

MGF: $\psi(t) = E(e^{tY}) = \int_{-\infty}^{\infty} e^{ty} dF(y)$

$\psi(t)$ exists if $\int_{-\infty}^{\infty} \mid e^{ty} \mid dF(y) \leq \infty$

Sometimes mgf does not exist.

Importance of MGF

Uniqueness Theorem: If two random variables have the same mgf, they have the same cdf except possibly at a countable number of points having 0 probability.

Continuity Theorem: Let $\{X_n\}$ and X have mgf $\{\psi_n(t)\}$ and $\psi(t)$ with $cdf's$ $F_n(x)$ and $F(x)$. Then a necessary and sufficient condition for $\lim_{n\to\infty} F_n(X) = F(X)$ is that for every t, $\lim_{n\to\infty} \psi_n(t) = \psi(t)$, where $\psi(t)$ is continuous at $t = 0$.

Inversion Formula: Knowledge of $\psi(t)$ enables the pdf or frequency function to be calculated.

Convolution Theorem: If Y_i are independent with mgf $\psi_i(t)$, then the mgf of $S = \sum_1^n Y_i$ is $\psi_S(t) = \prod_1^n \psi_i(t)$

If $\psi_i(t) = \psi(t) \quad \Rightarrow \quad \psi_S(t) = \psi(t)^n$

MOMENT GENERATING FUNCTIONS DO NOT ALWAYS EXIST!

For that reason one ordinarily uses a characteristic function of a distribution rather than the mgf.

Def. The characteristic function of a random variable y is

$$\varphi(t) = E(e^{ity}) = \int_{-\infty}^{\infty} e^{ity} dF(y) \text{ for } -\infty < t < \infty$$

$$|\varphi(t)| = \left|E(e^{ity})\right| \leq \int_{-\infty}^{\infty} \left|e^{ity}\right| dF(y) = \int_{-\infty}^{\infty} dF(y) = 1$$

as $\left|e^{ity}\right| = 1$.

$\Rightarrow$ Characteristic Functions always exist.

Relation between mgf and cf.

$$\varphi(t) = \psi(it)$$

$$\varphi(t) = \sum \frac{(it)^n}{n!} m_n \qquad \varphi^{(r)}(0) = i^r m_r$$

The characteristic function is a Fourier Transform; i.e. for any function $g(y)$

$$F.T.(q(y)) = \int_{-\infty}^{\infty} e^{ity} q(y) dy \qquad \text{if } q(y) \text{ is } pdf \;\Rightarrow\; c.f.$$

RELATION TO LaPLACE TRANSFORMS

Let Y be a non-negative r.v. with *pdf* $f(y)$; i.e. $P\{Y \geq 0\} = 1$. Then it is common to use Laplace Transforms instead of characteristic functions.

Def.: $s = a + ib \;\; (a > 0)$. Then the Laplace Transform of $f(y)$ is

$$f^*(s) = \int_0^\infty e^{-sy} f(y) dy$$

Since $e^{-sy} f(y) = e^{-iby} e^{-ay} f(y)$
The Laplace Transform is the equivalent of taking the Fourier Transform of $e^{-ay} f(y)$

Ex. Exponential $\varphi(t) = (1 - \frac{it}{\lambda})^{-1}$

$$f^*(s) = \int_0^\infty e^{-sy} \lambda e^{-\lambda y} dy = \frac{\lambda}{\lambda + s} = (1 + \frac{s}{\lambda})^{-1}$$

MOMENT GENERATING FUNCTIONS AND CHARACTERISTIC FUNCTIONS

	mgf	cf
Bernoulli	$(pe^t + q)$	$(pe^{it} + q)$
Binomial	$(pe^t + q)^n$	$(pe^{it} + q)^n$
Poisson	$e^{\lambda(e^{t-1})}$	$e^{\lambda(e^{it}-1)}$
Geometric	$pe^t/(1 - qe^t)$	$pe^{it}/(1 - qe^{it})$
Uniform over (a, b)	$\frac{e^{tb}-e^{ta}}{t(b-a)}$	$\dfrac{e^{itb} - e^{ita}}{it(b - a)}$
Normal	$e^{tm+\frac{1}{2}\sigma^2 t^2}$	$e^{itm-\frac{1}{2}\sigma^2 t^2}$
Exponential	$\frac{\lambda}{\lambda-t} = (1 - \frac{t}{\lambda})^{-1}$	$(1 - \dfrac{it}{\lambda})^{-1}$

INVERSION THEOREM

Integer valued R.V.

Let $Y = 0, \pm 1, \pm 2, \ldots$ with prob. $f(j) = P(Y = j)$

cf $\varphi(t) = E(e^{itY}) = \sum_{-\infty}^{\infty} e^{itj} f(j)$

Inversion Formula:

$$f(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \varphi(t) dt$$

$$f(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\left[\sum_{j=-\infty}^{\infty} e^{itj} f(j)\right] dt = \frac{1}{2\pi}\sum_{j=-\infty}^{\infty} f(j)\int_{-\pi}^{\pi} e^{i(j-k)t} dt$$

$$\begin{aligned}\int_{-\pi}^{\pi} e^{i(j-k)t} dt &= \int_{-\pi}^{\pi} [\cos(j-k)t + i\sin(j-k)t] dt \\ &= \Big|_{-\pi}^{\pi} \frac{\sin(j-k)t}{j-k} + i\Big|_{-\pi}^{\pi} - \frac{\cos(j-k)t}{(j-k)} \quad \text{for } j \neq k\end{aligned}$$

$\sin n\pi = 0$ for any integer n.

$\cos n\pi = \cos(-n\pi)$

$$\int_{-\pi}^{\pi} e^{i(j-k)t}\, dt = \begin{cases} 0 & \text{for } j \neq k \\ 2\pi & \text{for } j = k \end{cases}$$

INVERSION FORMULA FOR CONTINUOUS TYPE RANDOM VARIABLES

Let Y have $pdf\, f(y)$ and $cf\ \ \varphi(t)$ which is integrable; i.e.
$\displaystyle\int_{-\infty}^{\infty} |\varphi(t)|\, dt < \infty$

Inversion Formulae:

$$f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iyt}\varphi(t)dt$$

Ex. Normal Distribution (standard normal)

$$\varphi(t) = \int_{-\infty}^{\infty} e^{ity}\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy = e^{-t^2/2}$$

Replace t by $-t$

$$\int_{-\infty}^{\infty} e^{-ity} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy = e^{-t^2/2}$$

Interchange symbols t and y

$$\int_{-\infty}^{\infty} e^{-ity} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt = e^{-y^2/2}$$

Divide by $\sqrt{2\pi}$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ity} e^{-t^2/2} dt = \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

which is the inversion formular for $N(0, 1)$.

UNIQUENESS THEOREM

Let X be an arbitrary r.v. and let Y be $N(0,1)$ where X and Y are independent. Consider $Z = X + cY$ (c is a constant).

$$\varphi_Z(t) = \varphi_X(t)e^{-c^2t^2/2}$$

Note that $\varphi_Z(t)$ is integrable as $|\varphi_X(t)| \leq 1$

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itz}\varphi_z(t)dt$$

$$
\begin{aligned}
P(a < Z \le b) &= F_Z(b) - F_Z(a) = \int_a^b f(z)dz \\
&= \int_a^b \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it(x+cy)} \varphi_x(t) e^{-c^2t^2/2} dt dz \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_a^b e^{-itx} dx \right] \varphi_x(t) e^{-c^2t^2/2} e^{-itcy} dt
\end{aligned}
$$

Let $c \to 0 \Rightarrow z \to x$ and $dz \to dx$

$$
F_x(b) - F_x(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{e^{-ibt} - e^{-iat}}{-it} \right) \varphi_x(t) dt
$$

$\Rightarrow$ The distribution function is determined by its c.f.

$\Rightarrow$ If two r.v.'s have same characteristic function they have the same distribution function.

Examples

Double Exponential $f(x) = \frac{1}{2} e^{-|x|} \qquad -\infty < x < \infty$

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{2} e^{-|x|} dx = \frac{1}{2} \int_{-\infty}^{\infty} [\cos tx + i \sin tx] e^{-|x|} dx$$

Since $\sin(-tx) = -\sin tx$ and $\cos(-tx) = \cos(tx)$

$$\varphi(t) = \frac{1}{2} \int_{-\infty}^{\infty} (\cos tx) e^{-|x|} dx = \int_{0}^{\infty} (\cos \; tx) e^{-x} dx$$

$$\varphi(t) = \Big|_{0}^{\infty} \; \frac{e^{-x}[t \sin tx - \cos tx]}{1+t^2} = \frac{1}{1+t^2}$$

as $\sin 0 = 0, \quad \cos 0 = 1$

Double Exponential is sometimes called Laplace's Distribution.

Ex. What is $c.f.$ of the Cauchy distribution which has
$pdf \quad f(y) = \frac{1}{\pi(1+y^2)} \qquad -\infty < y < \infty$?

Note that by the inversion formula for the double exponential

$$\frac{1}{2}e^{|y|} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ity}\frac{1}{1+t^2}dt$$

Hence for the Cauchy distribution

$$\varphi(t) = \int_{-\infty}^{\infty} e^{ity}\frac{1}{\pi(1+y^2)}dy = e^{-|t|}$$

PROBLEMS

1. Let Y be any r.v.
   (a) Show $\varphi_y(t) = E[\cos tY] + iE[\sin tY]$
   (b) Show $\varphi_{-y}(t) = E[\cos tY] - iE[\sin tY]$
   (c) Show $\varphi_{-y}(t) = \varphi_y(-t)$

2. Let Y have a symmetric distribution around 0, i.e. $f(y) = f(-y)$
   (a) Show $E(\sin tY) = 0$ and $\varphi_y(t)$ is real valued.
   (b) Show $\varphi_y(-t) = \varphi_y(t)$

3. Let X and Y be ind. ident. dist. r.v.'s. Show $\varphi_{X-Y}(t) = |\varphi_x(t)|^2$

4. Let $Y_j \ \ (j = 1, 2)$ be independent exponential r.v.'s with $E(Y_j) = 1/\lambda_j$.

Consider $S = Y_1 + Y_2$

(a) Find the c.f. of S

(b) Using (i) and the inversion theorem find the pdf of S. Hint: If $f(y)$ is exponential with parameter $\lambda \ \ (E(Y) = 1/\lambda)$, then $\varphi(t) = \frac{\lambda}{\lambda - it}$ and $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ity}\frac{\lambda}{\lambda - it}dt = \lambda e^{-\lambda y}$ by the inversion formula.

2.7 Notes on Partial Fractions

Suppose $N^*(s)$ and $D^*(s)$ are polynomials in s. Suppose $D^*(s)$ is a polynomial of degree k and $N^*(s)$ has degree $< k$.

$$D^*(s) = \prod_{i=1}^{k}(s - s_i) \quad \text{(Roots are distinct)}$$

and roots of $N^*(s)$ do not coincide with $D^{*(s)}$. Then

$$\boxed{\frac{N^*(s)}{D^*(s)} = \sum_{i=1}^{k} \frac{A_i}{s - s_i}}$$

where constants A_i are to be determined.

Since $\mathcal{L}\left\{e^{-\lambda t}\right\} = (\lambda + s)^{-1}, \quad e^{-\lambda t} = \mathcal{L}^{-1}\{(\lambda + s)^{-1}\}$

$$\Rightarrow \quad \boxed{\mathcal{L}^{-1}\left\{\frac{N^*(s)}{D^*(s)}\right\} = \sum_{i=1}^{k} A_i e^{-s_i t}}$$

$$\frac{N^*(s)}{D^*(s)} = \sum_{i=1}^{k} \frac{A_i}{s - s_i}$$

$$\frac{N^*(s)}{D^*(s)} = \frac{\sum_{i=1}^{k} A_i \prod_{j \neq i}(s - s_j)}{\prod_1^k (s - s_i)}$$

$$N^*(s) = \sum_{i=1}^{k} A_i \prod_{j \neq i} (s - s_j)$$

Substitute $s = s_r$

$$N^*(s_r) = A_r \prod_{j \neq r} (s_r - s_j)$$

$$A_r = \frac{N^*(s_r)}{\prod_{j \neq r}(s_r - s_j)} \qquad r = 1, 2, \ldots, k$$

Since

$$
\begin{aligned}
D^*(s) &= \prod_{i=1}^{k}(s - s_i) \\
\frac{dD^*(s)}{ds} &= D'^*(s) = \sum_{i=1}^{k}\prod_{j\neq i}(s - s_j) \\
D'^*(s_r) &= \prod_{j\neq r}(s_r - s_j)
\end{aligned}
$$

$$
\boxed{A_r = \frac{N^*(s_r)}{D'^*(s_r)}} \qquad r = 1, 2, \ldots, k
$$

Example:

Let T_i be independent r.v. having pdf $q_i(t) = \lambda_i e^{-\lambda_i t}$ $(\lambda_1 \neq \lambda_2)$ and consider $T = T_1 + T_2$.

$$
\begin{aligned}
q_i^*(s) &= \lambda_i/(\lambda_i + s) \quad i = 1, 2 \\
q_T^*(s) &= q_1^*(s) q_2^*(s) = \lambda_1 \lambda_2 / (\lambda_1 + s)(\lambda_2 + s) \\
&= \frac{A_1}{(\lambda_1 + s)} + \frac{A_2}{(\lambda_2 + s)}
\end{aligned}
$$

$$N^*(s) = \lambda_1 \lambda_2 \quad , \quad D^*(s) = (\lambda_1 + s)(\lambda_2 + s), s_i = -\lambda_i$$

$$A_r = \frac{N^*(s_r)}{D'^*(s_r)} \quad , \quad D'^*(s) = (\lambda_1 + s) + (\lambda_2 + s)$$

$$A_1 = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \quad , \quad A_2 = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}$$

$$q_T^*(s) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left[\frac{1}{s + \lambda_1} - \frac{1}{s + \lambda_2} \right]$$

Taking inverse transforms gives

$$q(t) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left[e^{-\lambda_1 t} - e^{-\lambda_2 t} \right]$$

Multiple Roots

If $D^*(s)$ has multiple roots, the methods for finding the partial fraction decomposition is more complicated. It is easier to use direct methods.

Example: $N^*(s) = 1, \;\; D^*(s) = \;\; (a+s)^2(b+s)$

$$\frac{N^*}{D^*(s)} = \frac{1}{(a+s)^2(b+s)} = \frac{A_1}{a+s} + \frac{A_2}{(a+s)^2} + \frac{B}{b+s}$$

$$\frac{1}{(a+s)^2(b+s)} = \frac{A_1(a+s)(b+s) + A_2(b+s) + B(a+s)^2}{(a+s)^2(b+s)}$$

$$1 = A_1(a+s)(b+s) + A_2(b+s) + B(a+s)^2 \qquad (1)$$

$$1 = s^2[A_1 + B] + s[A_1(a+b) + A_2 + B(2a)] + [abA_1 + A_2 b + Ba^2]$$

$$A_1 + B = 0, \;\; A_1(a+b) + A_2 + 2aB = 0$$

$$A_1(ab) + A_2 b + Ba^2 = 1$$

The above three equations result in solutions for (A_1, A_2, B). Another way to solve for these constants is by substituting $s = -a$ and $s = -b$ in (1); i.e.

substituting $s = -b$ in (1)

$\Rightarrow\ B = (a-b)^{-2}\quad \Rightarrow A_1 = -B = -(a-b)^{-2}$

substituting $s = -a$ in (1) $\quad \Rightarrow\ A_2 = (b-a)^{-1}$

$$A_1 = -(a-b)^{-2}, A_2 = -(a-b)^{-1}, B = (a-b)^{-2}$$

$$\frac{N^*(s)}{D^*(s)} = \frac{1}{(s+a)^2(s+b)} = \frac{A_1}{s+a} + \frac{A_2}{(s+a)^2} + \frac{B}{s+b}$$

$$\mathcal{L}^{-1}\left\{\frac{N^*(s)}{D^*(s)}\right\} = -(a-b)^{-2}e^{-at} - (a-b)^{-1}te^{-at} + (a-b)^{-2}e^{-bt}$$

$$= \boxed{\frac{-e^{-at}}{(a-b)^2}[1 + t(a-b)] + \frac{e^{-bt}}{(a-b)^2}}$$

In general if $D^*(s) = (s - s_1)^{d_1}(s - s_2)^{d_2}...(s - s_k)^{d_k}$ the decomposition is

$$\frac{N^*(s)}{D^*(s)} = \frac{A_1}{s - s_1} + \frac{A_2}{(s - s_1)^2} + \ldots + \frac{A_{d_1}}{(s - s_1)^{d_1}}$$

$$+ \frac{B_1}{s - s_2} + \frac{B_2}{(s - s_2)^2} + \ldots + \frac{B_{d_2}}{(s - s_2)^{d_2}}$$

$$+ \ldots . +$$

$$+ \frac{Z_1}{(s - s_k)} + \frac{Z_2}{(s - s_k)^2} + \ldots + \frac{Z_{d_k}}{(s - s_k)^{d_k}}$$