The Mathematics of Ancient Alexandria and its Influence through the Arab Mathematicians to Modern Science

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As you know, the ancient Library of Alexandria was part of what was called the "Museum". This was not however a museum in the contemporary meaning of this word but rather a university, the first university in history. And one of its first professors was Euclid, known thoughout the world as "Euclid of Alexandria", the father of our mathematical tradition. A 13th century Arabic commentary says "There is no mathematician who has not walked in his footsteps". Indeed, Euclid's chief work, the Elements, is for us the first great landmark in the history of mathematical thought and its influence on subsequent scientific thought is immense.

I shall try in the following discussion to explain why this is so. So I begin with a most elementary but fundamental example: Euclid's proof of the fact that the set of prime numbers is infinite. In the following, by "number" I mean a positive integer. We recall that prime numbers are those numbers which are only divisible by themselves and unity. In a sense they are the "building blocks" in the realm of numbers, because all other numbers are composite, being built by taking products of primes. Even the most primitive examination reveals that the primes thin out among numbers as we proceed to larger numbers. So the question arises: do they stop somewhere? That is, is there a last prime, all numbers after that being composite? Euclid was the first to ask this question and the first to answer it, and in a most perfect fashion. Observe that no computer could ever answer the question, because it is a question about infinity. Only the mind could. Here is Euclid's proof. Suppose that on the contrary the set of prime numbers is finite, so we can count them in increasing order, omiting unity:

$$p_1, p_2, \cdots, p_n$$

Consider then the number:

$$M = \Pi + 1 \quad \text{were } \Pi \text{ is the product } p_1 p_2 \cdots p_n$$

This number being larger than the last prime, p_n, must be composite. Then M must have a prime factor, say q. So q must be one of of the

$$p_1, p_2, \cdots, p_n$$

But if $q = p_k$ for some $k = 1, ..., n$, then since q divides M as well the product Π it must divide their difference, which is unity. But this is absurd; for no number other than unity itself can divide unity, and we have omited unity. Consequently the contrary of our initial hypothesis must hold, that is, the set of prime numbers must be infinite.

Simple though the above proof is, it is still considered as one of the most elegant in all of mathematics. Consider the revolutions in the history of thought that this simple piece of mathematics contains: First, that the mind can pose a question about the infinite. Second, that the mind can answer that question in a definitive and final manner. Third, that the truth is found by showing that the opposite hypothesis leads to a contradiction. In fact all great proofs in mathematics from Euclid's time to our own have used Euclid's method of proof by contradiction.

Euclid's Elements consist of 13 books. The most advanced part of the Elements is the 5th book. Ancient commentators tell us that the theory expounded in this book, the "theory of proportion" as it was called, was the discovery of Eudoxus, a mathematician who was a contemporary of the great philosopher Aristotle. Euclid himself may have been his pupil. All his writings, like those of all the ancient Greek mathematicians preceding Euclid have perished. Nevertheless his work survived through Euclid. The theory of proportion was a theory of continuous magnitudes, like lengths, areas, volumes, time intervals, weights, etc.. So here we are talking about a theory of the outmost generality, which is most fundamental. To understand what was the issue that Eudoxus was addressing we must go back to the 6th and 5th centuries B.C. to what in the context of ancient Greek mathematics is prehistory, since from that time we have only fables. In the 6th century B.C. we have the mythical figure of Pythagoras, who declared that "all is number". When it comes to more concrete things like continuous magnitutes what this meant was that the relationship between two magnitudes of the same kind can be expressed as the ratio of two positive integers. In effect, if we are given say any two lengths a and a', we can find a suitably small length u such that a and a' are both integral multiples of u, that is, there exist positive integers m and m' such that:

$$a = mu \ \text{ and } \ a' = m'u$$

We then say that a and a' are "commensurable", u, which acts as a unit, being a common measure. The relationship of a to a' is then that of m to m', which is what defines the "rational number"

$$\frac{m}{m'}$$

However, in the 5th century B.C. the philosophy of the Pythagoreans, the followers of Pythagoras, suffered a devastating blow, when one of them, possibly Hippasus, reached a contradition when trying to find a common measure for the side and diagonal of a square or a regular pentagon. Then a crisis arose which almost brought mathematics to a standstill. The proofs of the most basic theorems in geometry, even that concerning similar triangles, on which the early proof of the celebrated Pythagorean theorem itself was probably based, all made use of the false assumption that all pairs of magnitudes of the same kind are commensurable. The crisis was overcome only when Eudoxus came up with his theory of proportions. Eudoxus made a great leap of abstraction when realizing that one should not try to give a direct definition of the notion of a continuous magnitude. One should instead proceed indirectly by stipulating that a magnitude can be multiplied by a positive integer to give another magnitude of the same kind and that two magnitudes of the same kind can be compared. Nor should one try to define directly what is meant by a relationship, or "proportion", between two magnitudes of the same kind, but rather only to define what is meant by equality of two proportions, say the proportion of a to a' being equal to the proportion of b to b', a and a' being magnitudes of the same kind, for example lengths, and also b and b' being magnitudes of the same kind, for example time intervals, but not necessarily of the same kind as the magnitudes of the first pair. The idea of Eudoxus was then the following. Consider first the case that could be dealt with earlier, namely the case that a and a' were commensurable. Then from the above there would exist positive integers m and m' such that

$$m'a = ma'$$

Similarly, if b and b' were commensurable there would exist positive integers n and n' such that

$$n'b = nb'$$

Then the equality of the proportion of a to a' to that of b to b' would simply be the equality of two rational numbers:

$$\frac{m}{m'} = \frac{n}{n'}$$

Consider now the general case when neither pair necessarily consists of commensurable magnitudes. Whatever pair (m, m') of positive integers we take, we either have $m'a \leq ma'$ or $m'a \geq ma'$ and the case of equality would never occur if the magnitudes a, a' were "incommensurable". Here is then Eudoxus' definition, the famous 5th definition of Euclid's 5th book:

If a, a' are magnitudes of the same kind, and b, b' are magnitudes of the same kind, we say that the proportion of a to a' is equal to the proportion of b to b' if for every pair (m, m') of positive integers, $m'a \leq ma'$ when and only when $m'b \leq mb'$, and $m'a \geq ma'$ when and only when $m'b \geq mb'$.

In the whole history of mathematics there is nothing involving a greater leap of abstraction. In fact, the content of Euclid's 5th book was not fully grasped until the 2nd half of the 19th century A.D. when the work of the German mathematician Dedekind appeared. Note that according the definition of Eudoxus any pair of magnitudes of the same kind divides the set of all pairs of positive integers (m, m') into two subsets N_1, N_2 according as to whether $m'a \leq ma'$ or $m'a \geq ma'$. The two subsets have common elements when and only when the magnitudes a, a' are commensurable. It is easy to see that this division actually corresponds to a division of the set of all rational numbers into two subsets Q_1 and Q_2, where Q_1 and Q_2 consists of the rationals of the form

$$\frac{m}{m'}$$

where (m, m') is from the subset N_1 and N_2 respectively. If q_1 is an element of Q_1 and q_2 is an element of Q_2 then we always have $q_1 \geq q_2$ and the two subsets Q_1 and Q_2 have a common element if and only if a and a' are commensurable. Dedekind then, following the above definition in Euclid's 5th book, actually identified a "proportion", a "real number" to use modern terms, with precisely such a division in the set of all rational numbers. And Dedekind's definition is the point of departure of modern analysis. That is why even though his own writings have all perished Eudoxus is counted among the greatest mathematicians in history.

I shall finally discuss Euclid's own most important contribution, what has since been universally known as "Euclidean Geometry". To properly assess the importance of this contribution, we must go back to the beginnings of human civilization which took place in this same land of Egypt and simultaneously in Mesopotamia. In these earliest civilizations, beginning at some time during the 3rd millenium B.C., empirical rules were found which provided answers to geometrical problems that arose in life. In Egypt, the repeated flooding of the Nile obliterated boundaries between pieces of land property and also decreased the area of some of the lots while increasing that of others. After each flooding the surveyors of the Pharao had to redraw the boundaries and also recalculate the area of each lot for the purpose of collecting taxes. This, according to ancient accounts, was the original stimulus for the discovery of empirical rules in plane geometry. On the other hand, the builders of the pyramids faced more challenging problems in solid geometry. The most impressive achievement of this empirical period of geometry in either Egypt or Mesopotamia is contained in a papyrus kept at the Egyptian Museum of Moscow. It gives the correct rule for the volume of a right pyramid of square base, thus similar to one of the Great Pyramids, in terms of its height and the length of each side of the base.

The ancient Greeks when they came to the scene of history in the 1st millenium B.C. developed close connections with Egypt. Thus the much older civilization of Pharaonic Egypt had a formative influence on ancient Greek civilization and the latter became the heir of the former. No such connections were developed with Mesopotamia, not only because of the greater distance, but also because the civilization of Mesopotamia, unlike that of Egypt, had suffered repeated devastating setbacks. Pythagoras is said to have spent more than 20 years studying in Egypt. When he eventually returned to Greece he brought with him the treasures of Egyptian empirical geometry. As I already mentioned, this earliest period of ancient Greek mathematics is enveloped in the mist of fable. Nevertheless it is certain that he and his followers gradually began to discover logical relationships between the different empirical facts. For example, the "theorem of Pythagoras" was found to be a logical consequence of facts concerning similar triangles. The fact that the sum of the angles of a triangle is equal to two right angles was found to be a logical consequence of properties of parallel lines. At some point in the 4th century B.C. the fantastic idea came up that all of geometry may derive from a few basic principles. But where to start, how to look for these basic principles? This must have been a problem of tremendous difficulty. For, geometry was something accessible to the senses as well as the mind. And it was almost impossible not to fall into the error of assuming, without explicitly stating so, something which was plainly obvious to the eye. Thus what seemed to be a logical consequence of the basic principles was in reality the result of tacit unconscious assumptions. Aristotle, the founder of systematic logic, in fact pointed out in his Prior Analytics that the theory of parallels then current contained a viscious circle of this type.

This is why it was a tremendous feat when Euclid finally succeeded in finding five postulates from which all of plane geometry could be derived by purely logical proceedures. Particularly impressive is the fact that he realized that Aristotle's objections could only be overcome by introducing his 5th postulate. His other four postulates are simple statements. His 1st postulate, for example, states that for any pair of points there is a single straight line segment joining these points. However his 5th postulate is not so simple. Here it is in its original formulation:

If a straight line falling on two other straight lines makes the sum of the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, shall meet on that side on which the said angles lie.

[Figure 1]

This is the most famous postulate in the history of mathematics. For two millenia a long series of very able mathematicians beleived Euclid's treatment to be deficient and tried to show that the 5th postulate can be derived from the other four. Finally, in the 1st half of the 19th century A.D. Euclid was vindicated when it was discovered that a new geometry is obtained when the 5th postulate is relaxed. This discovery of "non-Euclidean geometry" is one of the major turning points in the history of mathematics.

In summary, Euclid, by erecting geometry on a postulational basis, provided the first example of the hypothetico-deductive method. All major theories in the exact sciences that have been developed since have followed his example.

[Images]

These are images of the front and back side of the Fields Medal. Every four years the International Congress of Mathematicians is held in a different city around the world. I wish that at sometime in the near future Alexandria is chosen as the host city. It would be most appropriate. As the central event of the congress, the Fields Medal is awarded to between two and four mathematicians below the age of 40 who have made the most important contributions to mathematics. Despite this age limitation the Fields Medal is considered to be the Nobel Prize of mathematics. In fact it is held in even higher regard among mathematicians than is the Nobel Prize among physicists. This is because the Nobel Prize is decided by the Royal Swedish Academy and awarded by the King of Sweden, in a ceremony with only a limited number of guests invited, while the Fields Medal is decided by the International Mathematical Union and awarded by the Head of State of the country in which the congress is held, in the presence of about 10 thousand mathematicians from all over the world.

The front side of the Fields Medal depicts the greatest of mathematicians, Archimedes, his name in Greek inscribed on the right side. This man was once a student in exactly this place were we are now, listening attentively to the lessons of his teacher, Euclid. The inscription in Latin means:

> "Rise above oneself and grasp the world"

The back side of the Fields Medal depicts the diagram that he himself had asked to be placed on his grave, a sphere inscribed in a cylinder. I shall explain this in the following. The inscription in Latin means:

> "Awarded for mathematicians gathered from the entire world for their outstanding writings"

Archimedes spent his younger days studying here in Alexandria. It was here that he made his first great mathematical discoveries. He also started a life-long friendship with Conon, who later became Astronomer Royal of Egypt. After returning to his native city Syracuse, in Sicily, he still kept close relations with the scholars in Alexandria, till the end of his life. In those days if you were a mathematician and wanted to your work to be read by others, you sent it to the Library of Alexandria.

We have from ancient writers interesting particulars which throw light on his personality. He was a gentle, playful person, the epitome of the absent minded professor, being entirely lost in his thoughts. In those days people took long baths after which they annointed themselves with oil. After the bath Archimedes was lost for hours drawing figures on his own oiled skin. Once, when the solution to a problem on which he had been working for a long time came to him suddenly, he interruped his bath and ran out into the streets naked shouting “I found it, I found it!”. However the ultimate in this respect was his death. His native city was being sacked by a Roman army. There was death and destruction everywhere. But despite all this tumult Archimedes, now a very old man, was sitting in front of his diagrams thinking. When a soldier came up to him and stepped on his diagrams, Archimedes exclaimed: “Do not disturb my circles”. So the soldier in a rage slew him.

I now come to the sphere and the cylinder. The sphere is of course the simplest of curved surfaces, being the boundary of the ball, the simplest of solids. However, despite the fact that great mathematicians had preceeded Archimedes, no one had been able to find the surface area of a sphere. In mathematics, then as now, the most difficult problems are the simplest, which have been long standing. Archimedes devoted two books entitled “On the Sphere and the Cylinder, Books 1 and 2” to the geometry of the sphere. The last two but one propositions of the first book give the solution of a more general problem, namely that of finding the surface area of a segment of a sphere cut off by an arbitrary plane.

[Figure 2]

Consider a straight line from the center O of the sphere, which is perpendicular to the disk which is the base of the spherical segment. This will meet the sphere at a point V. All points on the circle which is the boundary of the base disk have the same distance L from V. The theorem of Archimedes simply tells us that the surface area of the spherical segment is equal to that of a disk of radius L. If the segment corresponds to a sector of angle θ and R is the radius of the sphere we have

$$L = 2R\sin\left(\frac{\theta}{2}\right)$$

and the area of the disk of radius L is

$$\pi L^2 = 4\pi R^2 \sin^2\left(\frac{\theta}{2}\right) = 2\pi R^2(1-\cos\theta)$$

therefore the theorem states that:

$$\text{Area of Spherical Segment} = 2\pi R^2(1-\cos\theta)$$

I shall now give an outline of how he derived this result. He considers a plane containing the straight line segment OV. Such a plane cuts the sphere in a circle and the base disk of the spherical segment in a straight line segment perpendicular to OV. The trace of the spherical segment on the plane in question is the segment of the circle cut off by the straight line segment. Archimedes then inscribes in this circular segment a polygon of $2n$ equal sides. He then considers the surface of revolution generated from the polygon by rotating the plane about the line OV. This surface consists of conical segments each obtained by the revolution of an edge of the polygon. There are n such conical segments.

[Figure 3]

Now the surface area of a conical segment generated by rotating a straight line segment, here an edge of the polygon, of length s, about an axis, here OV, is

$$A = 2\pi s \cdot \frac{1}{2}(r_1 + r_2)$$

where r_1 and r_2 are the distances of the end points of the segment from the axis of revolution. Here, for the mth conical segment, $m = 1, ..., n$,

$$s = 2R\sin\left(\frac{\theta}{2n}\right), \quad r_1 = R\sin\left(\frac{m\theta}{n}\right), \quad r_2 = R\sin\left(\frac{(m-1)\theta}{n}\right)$$

Hence, A_m, the area of the mth conical segment, is:

$$A_m = 2\pi R^2 \cdot 2\sin\left(\frac{\theta}{2n}\right) \cdot \frac{1}{2}\left[\sin\left(\frac{m\theta}{n}\right) + \sin\left(\frac{(m-1)\theta}{n}\right)\right]$$

Archimedes then shows that the area of the surface of revolution generated by the inscribed polygon is:

$$\sum_{m=1}^{n} A_m = 2\pi R^2 \left(S_n + \sin\left(\frac{\theta}{2n}\right)\sin\theta\right)$$

This gives a lower bound for the area of the spherical segment. Here S_n is the series:

$$S_n = \sum_{m=1}^{n-1} 2\sin\left(\frac{\theta}{2n}\right)\sin\left(\frac{m\theta}{n}\right)$$

Archimedes then shows that this is equal to:

$$S_n = \cos\left(\frac{\theta}{2n}\right)(1-\cos\theta) - \sin\left(\frac{\theta}{2n}\right)\sin\theta$$

Taking the limit $n \to \infty$, we have:

$$\sin\left(\frac{\theta}{2n}\right) \to 0, \quad \cos\left(\frac{\theta}{2n}\right) \to 1$$

therefore

$$S_n \to 1 - \cos\theta$$

and

$$\sum_{m=1}^{n} A_m \to 2\pi R^2(1-\cos\theta)$$

this is then the area of the spherical segment and the theorem is established.

I have simplified things somewhat. Archimedes' treatment is actually more rigorous. For, he also considers the corresponding circumscribed polygon whose sides are parallel to those of the inscribed one. The area of the surface of revolution of the circumscribed polygon, which he obtains in a similar manner, then gives an upper bound for the area of the spherical segment. Moreover, considering different values of n, he shows that each of the upper bounds exceeds each of the lower bounds, and that the difference between the two bounds corresponding to the same value of n tends to zero as $n \to \infty$. This clinches the proof. Observe that the sum S_n represents in the limit $n \to \infty$ what we now denote as the integral:

$$\int_0^\theta \sin\theta' d\theta'$$

so that the result of Archimedes above is equivalent to our formula:

$$\int_0^\theta \sin\theta' d\theta' = 1 - \cos\theta$$

The works of Archimedes contain a great number of theorems on the areas of plane domains, the volumes of solids and the areas of their boundary surfaces, and the centers of gravity of plane domains and solids. The concept of center of gravity is a concept that he himself introduced. In all these, like in the case of the surface area of a spherical segment, he introduces methods which anticipate those of the integral calculus by nearly two millenia.

I do not have time to discuss Archimedes' most impressive work, "On Floating Bodies, Book 2", which studies in detail the positions of equiliblium and their stability of a solid of a given density in the shape of a paraboloid of revolution cut off by a plane perpendicular to the axis, floating in a liquid of higher density.

Let me finally come to the Arabic contribution to mathematics. As everybody knows the Arab mathematicians were instrumental in preseving the works of the Hellenistic mathematicians. Everybody also knows that the chief original contribution of the Arab mathematicians was their invention of Algebra. But maybe not everyone fully understands how important this invention was. Well, in his original writings Archimedes presented his results only in words and diagrams because this was the only mode of mathematical expression available in his time. However, as we have seen, his ideas where very far ahead of his time and his results where very sophisticated. It was then nothing short of a miracle how the Arabs, when the Hellenistic manuscipts became available to them, were able to understand their content, with no one around to teach them. By their invention of Algebra they were able to express these works in a simpler and easier to grasp fashion. In this way many more mathematicians were able to contribute to the progress of their science. We should not forget that the tremendous progress which we have made in recent centuries is chiefly due to the fact that progress is now made through the synthesis of the contributions of a large number of mathematicians, not a few isolated geniuses. And this we owe to the Algebra invented by the Arabs. In addition, this invention opened up new horizons. For, a lot more can be expressed in algebraic terms than can be visualized geometrically. A obvious example is spaces of dimension greater than 3. In contemporary mathematics we in fact study infinite dimensional spaces. All this would have been impossible without the invention of the Arab mathematicians.